ON A LINEAR GROUP ISOMORPHIC TO $\mathbb{C}_2 \ltimes \mathbb{C}_m^{n-1}$

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Abstract. This paper focuses on a set of $n \times n$ matrices under modulo m, $\mathcal{S}_{n,m}$, and a group, $\mathfrak{T}_{n,m}$, generated by $\mathcal{S}_{n,m}$. $\mathcal{S}_{n,m}$ have the following properties: 1. the matrices are involutory; 2. any pair of matrices has order m; 3. any triplet of matrices has order 2. This paper also explores the expression of elements in $\mathfrak{T}_{n,m}$ and the geometrical significance of the group.

Contents

1. Notations

For the interest of readability, this paper uses the following set of notations.

$\mathbf{v}_n$	column vector which every element is v with dimension n
$C_{i,j}$	(column matrix) square matrix with ith and the jth column are $\mathbf{1}$, other columns are $\mathbf{0}$
I	the indentity matrix
O	zero matrix
$\langle g \rangle$	group generated by g
$\mathrm{Card}(\mathcal{S})$	number of elements in the set $\mathcal{S}$
$\lvert\mathfrak{G}\rvert$	order of group $\mathfrak{G}$

Unless otherwise stated, a, b, c, m, n, p, etc. are positive integers. Capital letters A_1, A_2, ..., B_1, B_2, ..., U_n, V_n are matrices after row and column exchanges of K (see definition 3.1).

In some cases, this paper would use capital letters C_1, C_2, ... to denote matrices in the form of $C_{i,j}$ defined above for $n \times n$ matrices, given a specific n.

Furthermore, kindly note that this paper will use LHS and RHS as abbrevation of left-hand side and right-hand side correspondingly in equations.

2. Introduction

2.1. **The settings.** This paper focuses on a special set of matrices, $\mathcal{S}_n$, which has the following properties (see note 3.9 for the notation of $\mathcal{S}$):

1. If $A \in \mathcal{S}_n$, A is a $n \times n$ square matrix.
2. If $A \in \mathcal{S}$, $A^2 = I$.
3. If $A, B \in \mathcal{S}_{n,m}$ (see note 3.9), $(AB)^m = I$.
4. If $A, B, C \in \mathcal{S}$, $(ABC)^2 = I$.

Group $\mathfrak{T}_{n,m}$ is generated by matrices in $\mathcal{S}_{n,m}$. This paper will prove the general form of the elements in $\mathfrak{T}_{n,m}$.

This paper draw inspiration from the study of triads transformations. In the papers of Crans *et. al.* and Fiore *et. al.* (see [1] and [2]), musical clock and an inscribed triangle is used to express the chords transformations. This paper extends the scene into a pure algebraic setting, in which the set and the group studied can be seen as the reflection of an inscribed polygon in a circle with nodes averagely arranged on it. This will be shown a little in section 5.

2.2. **The outline of this paper.** This paper can be viwed in two parts. The first part (Section 3) discusses the set of matrices ($\mathcal{S}$) concerned in this paper. We would provide the set of matrices that satisfies the three propositions mentioned above by giving the proofs. Furthermore, we would discuss some properties of 2-pairs and 3-pairs of matrices, which proved to be useful in the following discussions. The second part (Section 4) discusses the group ($\mathfrak{T}$) generated by $\mathcal{S}$. We will give the expression of elements in the group. Furthermore, the center of the group will be given.

3. The set $\mathcal{S}_{n,m}$

This section discusses a set of matrices ($\mathcal{S}$) which has special propositions. Specifically, all the elements in $\mathcal{S}$ are matrices after row and column exchanges of an original matrix. Their inversion is themselves. Any 2-pair AB where $A, B \in \mathcal{S}$ and $A \neq B$ has order n under modulo n. Any 3-pair ABC where $A, B, C \in \mathcal{S}$ and $A \neq B \neq C$ has order 2.

3.1. **The original matrix and the set of matrices ($\mathcal{S}$).**

Definition 3.1. For $n \times n$ matrices, the original matrix is defined as

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Remark 3.2. K can be expressed as follow:

$$K = C_{1,2} - I = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In the upcoming paragraphs, we will frequently refer to 3.2 for simplicity.

Lemma 3.3. *The product of n column matrices is the scalar multiplication of a power of 2 and the last matrix. Mathematically,*

$$C_1 C_2 C_3 \dots C_n = 2^{n-1} C_n, \tag{3.4}$$

where C_k $(1 \leq k \leq n)$ is in the form of $C_{i,j}$.

One can easily verify the above lemma.

Note 3.5. Specifically, $C_{i,j}^n = 2^{n-1}C_{i,j}$.

Proposition 3.6. *For $n \times n$ matrices, there exist $\frac{n(n-1)}{2}$ matrices, obtained by performing row and column exchanges on K, that are equal to their inverse.*

Proof. If A is a matrix that satisfies the above explanation, the proposition can be rewritten as $A^2 = I$.

For any integer pair (i, j) where $1 \leq i, j \leq n$ and $i \neq j$, $C_{i,j}$ will remain the same after row exchange. According to 3.3,

$$(C_{i,j} - I)(C_{i,j} - I) = C_{i,j}^2 - 2C_{i,j} + I = 2C_{i,j} - 2C_{i,j} + I = I$$

By row and column exchanges, there always exists a way which can turn the identy matrix into itself. After these exchanges, $C_{1,2}$ will turn into $C_{i,j}$ described above. Hence there are $\binom{n}{k}$ different pairs of i, j, which corresponds to $\frac{n(n-1)}{2}$ different matrices. □

Definition 3.7. Set $\mathcal{S}_n$ contains and only contains all $n \times n$ matrices of the form of $C_{i,j} - I$.

Note 3.8. $\text{Card}(\mathcal{S}_n) = \frac{n(n-1)}{2}$ as a direct result from proposition 3.6.

Note 3.9. We will use $\mathcal{S}$ to generally denote the matrices in any given n where $n \geq 3$. We will use $\mathcal{S}_{n,m}$ to denote $\mathcal{S}_n$ under modulo m.

Example 3.10. When $n = 3$, the three matrices are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 3.11. When $n = 4$, the six matrices are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix} \quad A_5 = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad A_6 = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

3.2. **Properties for 2-pairs.**

Proposition 3.12. *Every 2-pair, A_1A_2, where $A_1, A_2 \in \mathcal{S}_{n,m}$, satisfies*

$$(A_1A_2)^m = I \tag{3.13}$$

Proof. Firstly, we observe that $(C_{i_1,j_1} - C_{i_2,j_2})^k = O$ for any $1 \leq i_1, i_2, j_1, j_2 \leq m$ and $k \geq 2$.

Then we rewrite A_1 and A_2 into the form 3.2:

$$\begin{aligned}(A_1A_2)^n &= ((C_1-I)(C_2-I))^n \\ &= (C_1C_2-C_1-C_2+I)^n \\ &= (2C_2-C_1-C_2+I)^n \\ &= ((C_2-C_1)+I)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}(C_2-C_1)^k \\ &= \binom{n}{n-1}(C_2-C_1)+I \\ &= n(C_2-C_1)+I,\end{aligned}$$

which obviously equals to I under modulo n. □

Proposition 3.14. *Any two 2-pairs, A_1B_1 and A_2B_2, where $A_1, A_2, B_1, B_2 \in \mathcal{S}$, are commutative.*

Proof. We write A_1, A_2, B_1, B_2 into the form 3.2. Then, we examine the two products seperately.

$$\begin{aligned}(A_1B_1)(A_2B_2) &= ((C_1-I)(C_2-I))((C_3-I)(C_4-I)) \\ &= (C_1C_2-C_1-C_2+I)(C_3C_4-C_3-C_4+I) \\ &= (C_2-C_1+I)(C_4-C_3+I) \\ &= C_2C_4-C_2C_3+C_2-C_1C_4+C_1C_3-C_1+C_4-C_3+I \\ &= 2C_4-2C_3+C_2-2C_4+2C_3-C_1+C_4-C_3+I \\ &= C_2+C_4-C_1-C_3+I\end{aligned}$$

$$\begin{aligned}(A_2B_2)(A_1B_1) &= (C_4-C_3+I)(C_2-C_1+I) \\ &= C_4C_2-C_4C_1+C_4-C_3C_2+C_3C_1-C_3+C_2-C_1+I \\ &= 2C_2-2C_1+C_4-2C_2+2C_1-C_3+C_2-C_1+I \\ &= C_2+C_4-C_1-C_3+I\end{aligned}$$

The results are the same. The statement is proved. □

Proposition 3.14 yields the following.

Corollary 3.15. *Any powers of the 2-pair are commutable.*

Corollary 3.16. *A 2-pair does not necessarily commutes with a single matrix.*

Proof. The process is as same as which in proposition 3.14. We examine the two products seperately.

$$\begin{aligned}(C_1-I)((C_2-I)(C_3-I)) &= (C_1-I)(C_3-C_2+I) \\ &= C_1C_3-C_1C_2+C_1-C_3+C_2-I \\ &= 2C_3-2C_2+C_1-C_3+C_2-I \\ &= C_1-C_2+C_3-I\end{aligned}$$

$$\begin{aligned}((C_2-I)(C_3-I))(C_1-I) &= (C_3-C_2+I)(C_1-I) \\ &= 2C_1-C_3-2C_1+C_2+C_1-I \\ &= C_1+C_2-C_3-I\end{aligned}$$

These two results are not equal. □

Corollary 3.17. *The group generated by a single matrix and a 2-pair has the indentity as the only common element. Mathematically, if $A, B_1, B_2 \in \mathcal{S}_n$,*

$$\langle A \rangle \cap \langle B_1 B_2 \rangle = I.$$

Proof. For any C_1, C_2, C_3 in the form of $C_{i,j}$ and $C_1 \neq C_2 \neq C_3$, $C_1 \neq C_2 - C_3$. Hence $nC_1 + I = m(C_2 - C_3) + I$ if and only of $n = m$. □

Corollary 3.18. *Any combination of two different 2-pairs, (A_1B_1, A_2B_2), where $A_1B_1 \neq A_2B_2$, has the following proposition:*

$$\langle A_1 B_1 \rangle \cap \langle A_2 B_2 \rangle = I,$$

under a certain modulo.

Proof. For C_1, C_2, C_3, C_4 where $(C_1, C_2) \neq (C_3, C_4)$, $C_1 - C_2 \neq C_3 - C_4$. Therefore $n(C_1 - C_2) + I = m(C_3 - C_4) + I$ if and only if $n = m = 0$. Hence the corollary. □

3.3. **Properties for 3-pairs.**

Proposition 3.19. *Every 3-pair, $B = A_1A_2A_3$, where $A_1, A_2, A_3 \in \mathcal{S}$, satisfies $B^2 = I$.*

Proof. Write the three matrices into the form 3.2:

$$\begin{aligned}((C_1 - I)(C_2 - I)(C_3 - I))^2 &= (C_1C_2C_3 - C_1C_2 - C_1C_3 - C_2C_3 + C_1 + C_2 + C_3 - I)^2 \\ &= (4C_3 - 2C_2 - 4C_3 + C_1 + C_2 + C_3 - I)^2 \\ &= (C_1 - C_2 + C_3 - I)^2 \\ &= C_1^2 - C_1C_2 + C_1C_3 - C_1 - C_2C_1 + C_2^2 - C_2C_3 + C_2 + C_3C_1 - C_3C_2 \\ &+ C_3^2 - C_3 - C_1 + C_2 - C_3 + I \\ &= 2C_1 - 2C_2 + 2C_3 - C_1 - 2C_1 + 2C_2 - 2C_3 + C_2 + 2C_1 - 2C_2 \\ &+ 2C_3 - C_3 - C_1 + C_2 - C_3 + I \\ &= I.\end{aligned}$$

Hence the proposition. □

4. The group $\mathfrak{T}_{n,m}$ generated by elements in $\mathcal{S}_{n,m}$

This section elaborates on a group which is constructed by the matrices in $\mathcal{S}$. Firstly, we discuss a conjugation. Then, we will provide the general form of the elements in the group. Finally, we will discuss the order of the group using conclusions from combinatory.

4.1. **An important conjugation.**

Proposition 4.1. *For $A_1, A_2, B \in \mathcal{S}$, where $A_1 \neq A_2 \neq B$, the following conjugation exists:*

$$B(A_1A_2)B^{-1} = (A_1A_2)^{-1}. \tag{4.2}$$

Proof. Equation 4.2 is equivalent to the following:

$$\begin{aligned}B(A_1A_2)B^{-1} = (A_1A_2)^{-1} = A_2^{-1}A_1^{-1} &= A_2A_1 \\ BA_1A_2 &= A_2A_1B.\end{aligned}$$

We will now demonstrate the above equation by separately examining the products of the left-hand side and the right-hand side.

$$\begin{aligned}LHS &= (C_1 - I)(C_2 - I)(C_3 - I) \\ &= C_1 - C_2 + C_3 - I,\end{aligned}$$

$$
\begin{aligned}
RHS &= (C_3 - I)(C_2 - I)(C_1 - I) \\
&= C_1 - C_2 + C_3 - I.
\end{aligned}
$$

$LHS = RHS$. Hence the proposition. □

A more general form of proposition 4.1 is provided in the following corollary.

Corollary 4.3. *For $A, B \in \mathcal{S}$, where $A \neq B$, the following conjugation exists:*

$$A(AB)^m A^{-1} = (AB)^{-m}, \tag{4.4}$$

under certain modulo.

Proof. The proof is quite straightforward:

$$
\begin{aligned}
A(AB)^m A^{-1} &= AAB(AB)^{m-1}A^{-1} \\
&= B(AB)^{m-1}A^{-1} \\
&= (BA)B(AB)^{m-2}A^{-1} \\
&= (BA)^2 B(AB)^{m-3}A^{-1} \\
&\cdots \\
&= (BA)^{m-1}BA^{-1} \\
&= (BA)^{m-1}BA \\
&= (BA)^m \\
&= (AB)^{-m}
\end{aligned}
$$

□

Corollary 4.5. *For $U, V_1, V_2, \ldots, V_{n-1} \in \mathcal{S}_{n,m}$, and $A = U(UV_1)^{a_1}(UV_2)^{a_2} \cdots (UV_{n-1})^{a_{n-1}}$, $B = (UV_1)^{b_1}(UV_2)^{b_2} \cdots (UV_{n-1})^{b_{n-1}}$, where $a_i, b_i \in \mathbb{Z}_m$, the following conjugation exists:*

$$ABA^{-1} = B^{-1}. \tag{4.6}$$

Proof. We define $P_i = UV_i$. Firstly, we note that $A^{-1} = P_{n-1}^{a_{n-1}} P_{n-2}^{a_{n-2}} \cdots P_1^{a_1} U$. Hence

$$
\begin{aligned}
ABA^{-1} &= UP_1^{a_1}P_2^{a_2} \cdots P_{n-1}^{a_{n-1}} P_1^{b_1}P_2^{b_2} \cdots P_{n-1}^{b_{n-1}} P_{n-1}^{a_{n-1}} P_{n-2}^{a_{n-2}} \cdots P_1^{a_1} U \\
&= UP_1^{b_1}P_2^{b_2} \cdots P_{n-1}^{b_{n-1}} U = UP_1^{b_1}P_2^{b_2} \cdots P_{n-1}^{b_{n-1}} U^{-1} \\
&= UP_1^{b_1}U^{-1}UP_2^{b_2}U^{-1} \cdots UP_{n-1}^{b_{n-1}}U^{-1} = (UP_1^{b_1}U^{-1})(UP_2^{b_2}U^{-1}) \cdots (UP_{n-1}^{b_{n-1}}U^{-1}) \\
&= P_1^{-b_1}P_2^{-b_2} \cdots P_{n-1}^{-b_{n-1}} = P_{n-1}^{-b_{n-1}}P_{n-2}^{-b_{n-2}} \cdots P_1^{b_1} \\
&= B^{-1}.
\end{aligned}
$$

□

4.2. **Group $\mathfrak{T}$.**

Definition 4.7. For any $n \in \mathbb{N}^*$ and $n \geq 3$, group $\mathfrak{T}_n$ is generated by all the elements in $\mathcal{S}_n$.

Note 4.8. We use $\mathfrak{T}$ to generally refer to a group with an undetermined n, and we use $\mathfrak{T}_{n,m}$ to express $\mathfrak{T}_n$ under modulo m.

Now we focus on a matrix and a series of 2-pairs, $U, UV_1, UV_2, \ldots, UV_{n-1}$ where $U, V_1, V_2, \ldots, V_{n-1} \in \mathcal{S}_n$.

Theorem 4.9. *Every element in group $\mathfrak{T}_{n,m}$ can be uniquely expressed as*

$$U^{p_1}(UV_1)^{p_2}(UV_2)^{p_3} \cdots (UV_{n-1})^{p_n}, \tag{4.10}$$

where $p_1, p_2, \ldots, p_n \in \mathbb{Z}_m$, $U, V_1, V_2, \ldots, V_n \in \mathcal{S}_n$.

Proof will be presented later after the discussion of proposition 4.12.

Lemma 4.11. *Any three matrices, C_1, C_2, C_3, where C_k is in the form $C_{i,j}$, has the following proposition:*

$$(C_2 - C_1)(C_3 - C_1) = O$$

Proof. This can be proved by straightforward calculations. □

Proposition 4.12. *All the matrices in $\mathcal{S}_n$ $(n \geq 3)$ can be expressed by a specific combination of a matrix and $n-1$ 2-pairs under modulo m:*

$$U(UV_1)^{p_2}(UV_2)^{p_3}\cdots(UV_{n-1})^{p_n}, \tag{4.13}$$

where $p_2, \ldots, p_n \in \mathbb{Z}_m$, $U, V_1, V_2, \ldots, V_n \in \mathcal{S}_n$.

Proof. Now we consider a special set of matrices. We denote $C_{1,k+1}$ as C_k $(1 \leq l \leq n-1)$, and we denote $C_{n-1,n}$ as C_n, and $U = C_1 - I$, $V_i = C_{k-1} - I$[1]. Now we will prove any other $C_{i,j}$ can be expressed as equation 4.13.

As before, we write $U, V_1, \ldots, V_{n-1}$ into the expression contains C_i. We assume the matrix to be expressed can be written as $C_{a,b} - I$. Therefore, we have the following equation:

$$C_{a,b} - I = (C_1 - I)\left(\prod_{k=2}^{n}(p_k(C_k - C_1) + I)\right).$$

Note that in this equation we utilized the result from proposition 3.12. Using lemma 4.11, we can simplify the right-hand side of the equaiton above:

$$\begin{aligned}
C_{a,b} - I &= (C_1 - I)\left(\sum_{k=2}^{n} p_k C_k + \sum_{k=2}^{n} p_k C_1 + I\right) \\
&= \sum_{k=2}^{n} p_k C_1 C_k - \sum_{k=2}^{n} p_k C_k - \left(\sum_{k=2}^{n} p_k\right) C_1^2 + \left(\sum_{k=2}^{n} p_k\right) C_1 + C_1 - I \\
&= 2\sum_{k=2}^{n} p_k C_k - \sum_{k=2}^{n} p_k C_k - 1 - 2\left(\sum_{k=2}^{n} p_k\right) C_1 + \left(\sum_{k=2}^{n} p_k\right) C_1 + C_1 - I \\
&= \sum_{k=2}^{n} p_k C_k - \left(\sum_{k=2}^{n} p_k\right) C_1 + C_1 - I.
\end{aligned}$$

This induces

$$C_{a,b} = \sum_{k=2}^{n} p_k C_k - \left(\sum_{k=2}^{n} p_k\right) C_1 + C_1. \tag{4.14}$$

Now we use induction to prove the proposition. When $n = 3$, the three matrices are in example 3.10. In this case, $A_2 = A_1(A_1A_2)$, $A_3 = A_1(A_1A_3)$.

We now assume the proposition is true for $n = k$, and we prove it holds for $n = k+1$. If $n = k$ holds, this is to say any matrix in $\mathcal{S}_k$ can be expressed by k matrices in the above form, which means any $C_{i,j}$ where $1 \leq i, j \leq k$ and $i \neq j$ can be expressed. So we only have to prove any $C_{i,k+1}$ where $2 \leq i \leq k$ can be expressed as right-hand side in equation 4.14.

[1] I apologize for the confusing notation.

We write the right-hand side of equation 4.14 in a more intuitive way:

$$RHS = \begin{pmatrix} \sum_{i=2}^{k} p_i & 0 & p_2 & p_3 & \cdots & p_{k-1}+p_{k+1} & p_k+p_{k+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \sum_{i=2}^{k} p_i & 0 & p_2 & p_3 & \cdots & p_{k-1}+p_{k+1} & p_k+p_{k+1} \end{pmatrix} - \begin{pmatrix} \sum_{i=2}^{k+1} p_i & \sum_{i=2}^{k+1} p_i & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \sum_{i=2}^{k+1} p_i & \sum_{i=2}^{k+1} p_i & 0 & \cdots & 0 \end{pmatrix} + C_1$$

$$= \begin{pmatrix} -p_{k+1}+1 & -\sum_{i=2}^{k+1} p_i + 1 & p_2 & p_3 & \cdots & p_{k-1}+p_{k+1} & p_k+p_{k+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -p_{k+1}+1 & -\sum_{i=2}^{k+1} p_i + 1 & p_2 & p_3 & \cdots & p_{k-1}+p_{k+1} & p_k+p_{k+1} \end{pmatrix}.$$

According to our assumption, p_{k+1} must be 1. Therefore, the first column is $\mathbf{0}_{k+1}$. Now we consider ith column is $\mathbf{1}_{k+1}$ when $3 \leq i < k-1$. In this case, $p_k = -1$, $\sum_{i=2}^{k+1} p_i = -1$, the second column is $\mathbf{0}_{k+1}$. If the kth column is $\mathbf{1}_{k+1}$, than for all $2 \leq i \leq k$, $p_i = 0$, hence $\sum_{i=2}^{k+1} p_i = -1$, the second column is $\mathbf{0}_{k+1}$.

Thus the proposition holds. □

Note 4.15. In the following, we will refer the set contains $U, V_1, V_2, \ldots, V_{n-1}$ to as $\mathcal{G}_n$.

4.3. **Proof of theorem 4.9.**

Proof of Theorem 4.9. Initially, we will examine the special case involving words with length 1 and 2. Subsequently, we will apply recursion to demonstrate that other words can be represented in equation 4.10.

Firstly, note that every word does not have two consecutive matrices that are the same according to the definition of $\mathcal{S}$. According proposition 4.12, every word of length 1 can be expressed as 4.10.

Now consider words of lengths 2. For words in the form of V_aV_b where $a \neq b$ and $a, b \in \mathbb{Z}_n$, we can do the follwoing transformation:

$$V_aV_b = V_aIV_b = V_aUUV_b = (UV_a)^{-1}(UV_b) = (UV_a)^{m-1}(UV_b).$$

For words consist of a matrix $A \in \mathcal{S}_n - \mathcal{G}_n$ and shapes as AU, we can use the conjugation. We first denote $B = AU$, then

$$U^{-1}AU = A^{-1}.$$

Since 4.12, A^{-1} can be expressed into the form 4.10, hence $U^{-1}AU$ can, thus AU can.

Therefore, for any word, we can divide it into the form of the product of several words of length 1 and 2 in the form above, and it can be written into the form 4.10.

Fot the uniqueness, suppose for some $a_i, b_i \in \mathbb{Z}_m$ $(1 \leq i \leq n-1)$ and $m, n \in \{0, 1\}$, where $a_i \neq b_i$ for every i, exsits

$$U^m(UV_1)^{a_1}(UV_2)^{a_2}\cdots(UV_{n-1})^{a_{n-1}} = U^n(UV_1)^{b_1}(UV_2)^{b_2}\cdots(UV_{n-1})^{b_{n-1}}. \tag{4.16}$$

If $m = n$, then equation 4.16 can be simplified as

$$(UV_1)^{a_1}(UV_2)^{a_2}\cdots(UV_{n-1})^{a_{n-1}} = (UV_1)^{b_1}(UV_2)^{b_2}\cdots(UV_{n-1})^{b_{n-1}}.$$

This induces that

$$(UV_1)^{a_1-b_1}(UV_2)^{a_2-b_2}\cdots(UV_{n-1})^{a_{n-1}-b_{n-1}} = I.$$

Recall corollary 3.18, the above equation implies $a_1 - b_1 = a_2 - b_2 = \cdots = a_{n-1} - b_{n-1} = 0$, which contradicts to the assumption.

If $m \neq n$, then equation 4.16 can be simplify as

$$U(UV_1)^{a_1}(UV_2)^{a_2}\cdots(UV_{n-1})^{a_{n-1}} = (UV_1)^{b_1}(UV_2)^{b_2}\cdots(UV_{n-1})^{b_{n-1}}.$$

This implies that

$$(UV_1)^{a_1-b_1}(UV_2)^{a_2-b_2}\cdots(UV_{n-1})^{a_{n-1}-b_{n-1}} = U.$$

This is impossible because of 3.17. □

Corollary 4.17.

$$|\mathfrak{T}_{n,m}| = 2m^{n-1}.$$

Proof. This is a direct result of combinatory and equation 4.10. □

Proposition 4.18.

$$\mathfrak{T}_{n,m} \simeq \mathbb{C}_2 \ltimes \mathbb{C}_m^{n-1}. \tag{4.19}$$

Proof. We first claim that $\langle UV_1, UV_2, \ldots, UV_{n-1}\rangle$ is a normal subgroup of $\mathfrak{T}_{n,m}$. For $\forall g \in \mathfrak{T}_n$, by equation 4.10, we have

$$g = U^{p_1}(UV_1)^{p_2}(UV_2)^{p_3}\cdots(UV_{n-1})^{p_n}$$

and

$$g^{-1} = (UV_{n-1})^{m-p_n}(UV_{n-1})^{m-p_{n-1}}\cdots(UV_1)^{m-p_2}U^{p_1},$$

where $p_1 \in \{0,1\}$ and $p_2, p_3, \ldots, p_n \in \mathbb{Z}_m$. Now, assume $h \in \langle UV_1, UV_2, \ldots, UV_{n-1}\rangle$.

If $p_1 = 0$,

$$ghg^{-1} = h$$

because of the communitivity. And if $p_1 = 1$,

$$ghg^{-1} = h^{-1}$$

according to 4.5, which is obviously in $\langle UV_1, UV_2, \ldots, UV_{n-1}\rangle$.

Finally, by 3.17, we have

$$\mathfrak{T}_{n,m} = \langle U\rangle \ltimes \langle UV_1, UV_2, \ldots, UV_{n-1}\rangle \simeq \mathbb{C}_2 \ltimes \mathbb{C}_m^{n-1}.$$

□

Example 4.20. $\mathfrak{T}_{3,12}$ is construct by three matrices ([2]):

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad V = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad W = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The expression of elements in $\mathfrak{T}_{3,12}$ is

$$U^k(UV)^m(UW)^n,$$

where $k \in \{0,1\}$ and $m, n \in \mathbb{Z}_{12}$.

The order of $\mathfrak{T}_{3,12}$ is 288.

And

$$\mathfrak{T}_{3,12} \simeq \mathbb{C}_2 \ltimes (\mathbb{C}_{12} \times \mathbb{C}_{12}).$$

4.4. **Center of $\mathfrak{T}_{n,m}$.**

Proposition 4.21. *For $m \geq 3$, if n is even, $Z(\mathfrak{T}_{n,m})$ consists of all the elements in the following set*

$$\{(UV_1)^{\frac{n}{2}a_1}(UV_2)^{\frac{n}{2}a_2}\cdots(UV_{n-1})^{\frac{n}{2}a_{n-1}}\},$$

where $a_i \in \{0,1\}$.

If n is odd, $Z(\mathfrak{T}_{n,m})$ is a trivial group.

Proof. Recall that $Z(\mathfrak{T}_{n,m})$ is a subgroup of $\mathfrak{T}_{n,m}$ where for $\forall A \in Z(\mathfrak{T}_{n,m})$ and $\forall B \in \mathfrak{T}_{n,m}$, $AB = BA$. Now, consider a element $A \in \langle UV_1, UV_2, \ldots, UV_{n-1}\rangle$. $A \in Z(\mathfrak{T}_n)$ if and only if $AU = UA$. Thus

$$UAU^{-1} = A^{-1} = A,$$

$$A^2 = I.$$

As a result, A has to be in the set above to have this property. If $B = UA$, then we prove the following counterexample by contradction.

$$(UV_1) = (UA)(UV_1)(UA)^{-1} = U(A(UV_1)A^{-1})U^{-1} = U(UV_1)U^{-1} = (UV_1)^{-1},$$

but $UV_1 = (UV_1)^{-1}$ is impossible, as (UV_1) has order m.

Hence the proposition. □

5. Geometrical interpretation

Group $\mathfrak{T}_{n,m}$ has geometrical significance. This section will bridge the above sections and a geometrical shape (and its reflection).

Imagine a circle with m nodes averagely arranged on a circle marked from 0 to $m-1$, and we denote it $\mathcal{O}_m$. Now, a n-gon is inscribed in $\mathcal{O}_m$ ($n \leq m$) with every vertice on a node. We denote this n-gon using a vector, in which every element is the number on the node corresponding to the vertice. This n-gon can reflect among one of the midperpendicular line of a line segment between its two vertices.

Mathematically, reflection of node x with the axis of reflection is the midperpendicular of vertices a and b can be defined as

$$I_{a+b}(x) := -x + (a+b). \tag{5.1}$$

If we extend this notation into the n-gon vector, we have

$$I_{a+b}(\mathbf{x}) = (I_{a+b}(x_1), I_{a+b}(x_2), \ldots, I_{a+b}(x_n)), \tag{5.2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

A reflection can be seen as the left multiplication of a matrix in $\mathcal{S}_n$ of the vector denote a n-gon. Thus, the elements in $\mathfrak{T}_{n,m}$ can be seen as the possible transformation after a series of reflections.

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