On a Linear Group of Seventh Chord Transformations

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Abstract

Seventh chords are widely used in music. What are the relationships between different seventh chords? Can these relationships be explained mathematically? This paper classifies transformations between seventh chords into six types using Euler's principle, namely U_1, V_1, W_1, U_2, V_2, and W_2. Each of these transformations is in matrix form. This paper constructs a group generated by these transformations and names this group the General Transformation Group (GT). This paper claims that every element in GT can be expressed as ${U_1}^k(U_1V_1)^{a_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3}$. This paper also determines that GT is isomorphic to $\mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12})$, and its center is isomorphic to $C_2 \times C_2 \times C_2$. By using this group, we can examine the transformation between dominant sevenths and half-diminished sevenths from a generalized 4-chord perspective. This group also gives musicologists a theoretical guideline to seventh chords and their relationships, and provides musicians and auto accompaniment with a clear picture of the usage of seventh chords. This paper encourages further research into transformations of chords beyond seventh chords, aiming to broaden the applicability of the GT group's principles.

Contents

1 Introduction

We will discuss seventh chord transformations in a general way in this paper. We claim that there exist six transformations for seventh chords, and we will construct a group, namely the General Transformation Group (GT), using combinations of these transformations. In Section (3.1), we first make four observations on the properties of the six transformations, and then the structure of GT is presented with proofs. We will discuss the center of GT in Section (3.2), which is isomorphic to the internal direct product of three cyclic groups. Then, we will focus on a subgroup of GT that has crucial musical significance in Section (4). We will demonstrate that when being acted by two transformations repeatedly, a dominant seventh or a half-diminished seventh can form a trajectory that traverses all 24 dominant and half-diminished sevenths.

Chords push music forward. They can create tension and give nuance differences in music. Musicians in the Classical Era relied mainly on triads (which contain three notes) in their compositions. Into the 19th and 20th centuries, musicians began to use seventh chords (which consist of four notes) and other chords with more notes. Studies on chords also have a long history. Euler established a network for transformations between different major and minor triads[1]. Later, German musicologist Hugo Riemann constructed the PLR group based on the three transformations in Euler's work. 20th-century musicologists used the PLR group as a fundament for studying chords. Crans et al. demonstrated that the PLR group is a dihedral group of order 24 generated by L and R in their work ([3]). Fiore and Noll constructed a more general group for triad transformations using linear representations in ([5]). Fiore also generalized contextual transformations into $\mathbb{Z}_m$, where m is not necessarily 12 ([4]). Triad transformations have been seen in applications widely. Cohn discovered Beethoven applied a sequence of repeated transformations in his ninth Symphony ([2]) [2]. Wang in his book ([7]) mentioned Haydn applied a series of triads transformations in his Piano Sonata [3].

There are relatively few studies on seventh chords. Childs separated seventh chords into two classes that were interconvertible ([1]). These two classes were the dominant seventh chord and the half-diminished seventh chord, which in his paper is called the Tristan chord because of the famous use of it in the beginning of Wagner's Tristan und Isolde. XNP group for which we will discuss is a group of transformations for these two classes of chords. For an easier introduction to this topic, one can refer to Section (7.3) to (7.5) of Wang's book ([7]).

2 Basic Music Theory and Seventh Chord Transformations

In this section, we will first introduce some basic music theories that will be the fundamental of this paper in Section (2.1). Then we will define six transformations between seventh chords in both vector and matrix form in Section (2.2).

2.1 Basic music theory and seventh chords

Twelve equal temperament lies at the heart of modern music theory. This temperament states that for every two consecutive notes, the ratio of frequencies of the higher pitch to the lower pitch is constant, which is $\sqrt[12]{2}$, being set considering that the ratio of frequency between an octave is 2. In this way, pitch classes can be substituted using numbers in $\mathbb{Z}_{12}$. Musicologists also consider two basic actions in music, which are presented in the T/I group. This group is defined as

$$T_n(x) := x + n \qquad I_n(x) := -x + n \qquad x, n \in \mathbb{Z}_{12}.$$

[1]Euler named them as P (parallel), L (leading tone), and R (relative) according to their musical properties.

[2]Symphony No.9 in D minor, Op.125.

[3]Piano Sonata in E minor, Hob XIV: 34.

With the equal temperament and the T/I group, musicologists discovered a way to intuitively show a chord, which is the musical clock. A musical clock is a circle with twelve pitch classes averagely arranged on it. Moreover, as a chord consists of several pitch classes, it can be described as an inscribed polygon of the musical clock. Here we state without proof that every flip with the axis of flip as the mid-perpendicular of the line segment between the node m and n at the musical clock can be denoted as I_{m+n}.

There are two types of seventh chords for which we are concerned in this paper, the dominant seventh and the half-diminished seventh. Every seventh chord can be denoted using either a set or a four-tuple. A dominant seventh can be encoded as $\{x, x+4, x+7, x+10\} \subseteq \mathbb{Z}_{12}$ or as $(x, x+4, x+7, x+10) \in \mathbb{Z}_{12}{}^{\times 4}$. Similarly, a half-diminished seventh can be encoded as $\{x, x+3, x+6, x+10\} \subseteq \mathbb{Z}_{12}$ or as $(x, x+3, x+6, x+10) \in \mathbb{Z}_{12}{}^{\times 4}$. In this encoding, x is the *root* of a chord, $x+4$ or $x+3$ is the *third*, $x+7$ or $x+6$ is the *fifth*, and $x+10$ is the *seventh*. More generally, a seventh chord is a four-element set, $\{a, b, c, d\} \subseteq \mathbb{Z}_{12}$, or a four-tuple, $(a, b, c, d) \subseteq \mathbb{Z}_{12}{}^{\times 4}$. In this paper, we will focus on the tuple form, as different arrangings of these four notes suggest nuance differences between chords (though it is constructed using the same pitch classes).

2.2 Seventh chord transformations

Seventh chord transformations can be described as $G \times S \to S$ where G is the group of actions and S is the set of all seventh chords. Musicians seek a series of trajectories $(s_0, s_1, s_2, \ldots, s_n)$ within S via an associated sequence $(g_0, g_1, g_2, \ldots, g_n)$ of transformations $g_i \in G$ satisfying $g_i(s_{i-1}) = s_i$. For the interest of *logic* in music, two consecutive chords should be strongly related. This can be achieved by moving as little as possible the number of pitch classes in a chord. Therefore, we use the I transformations and the goal we made in Section (2.1) to construct tranformations. For each seventh chord (a, b, c, d), we have the following six transformations:

$$\begin{aligned}
I_{b+d}(a,b,c,d) &= (-a+b+d, d, -c+b+d, b) \\
I_{a+c}(a,b,c,d) &= (c, -b+a+c, a, -d+a+c) \\
I_{a+d}(a,b,c,d) &= (d, -b+a+d, -c+a+d, a) \\
I_{c+d}(a,b,c,d) &= (-a+c+d, -b+c+d, d, c) \\
I_{a+b}(a,b,c,d) &= (b, a, -c+a+b, -d+a+b) \\
I_{b+c}(a,b,c,d) &= (-a+b+c, c, b, -d+b+c).
\end{aligned} \tag{1}$$

We name these transformations from (1) as X, N, P, G, R, and B respectively.

From these transformations, we notice that every single transformation keeps two elements from the original tuple while changing another two. What is the relationship between the new seventh and the original one? When we are considering relationships between chords in music, the most common practice is to examine their root. Below is a list for the relationship of roots for a seventh chord after each of these transformations:

Transformation	Original Root	Final Root
X	x	$x+4$
N	x	$x+9$
P	x	x
G	x	$x+7$
R	x	$x+6$
B	x	$x+1$

Table 1: Relationships between roots after a transformation for dominant sevenths.

We can also use matrices to illustrate these transformations, as below:

$$U_1 = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix} \quad W_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \quad W_2 = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix},$$

where U_1, V_1, W_1, U_2, V_2, and W_2 stand for N, X, P, G, R, and B correspondingly. Later we will explain their naming. These transformations are *contextual determined*, which means that the specific transformation is determined by the chord intself. Here we demonstrate an example of how a transformation acts on a chord:

$$W_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} d \\ -b+a+d \\ -c+a+d \\ a \end{pmatrix}. \tag{2}$$

A seventh chord can have *inversions*. Inversions are permutaions of the original tuple. For example, the chord $(4,7,10,0)$ is a inversion of the chord $(0,4,7,10)$. Generally, for a seventh chord (a,b,c,d), it has four inversions: $(a,*,*,*)$ (*root position*), $(b,*,*,*)$ (*first inversion*), $(c,*,*,*)$ (*second inversion*), and $(d,*,*,*)$ (*third inversion*). The name of the inversions are based on the chord's root, which is the first entry of the tuple. Entries in $*$ are not important for the chord's inversion. In other words, they are not determinants. And for each of these inversions, when the six matrices are acting on them, the real impact are different in terms of changes in roots. The relation between roots can be classified into one of the six transformations (P, X, N, G, B, and R). Therefore, each of these matrices are different transformations when comes into different inversions. The correspondence is listed below for four basic inversions.

Transformation	Root Position	First Inversion	Second Inversion	Third Inversion
U_1	X	P	X	P
V_1	N	R	B	G
W_1	P	X	P	X
U_2	G	N	N	B
V_2	R	G	P	R
W_2	B	B	G	N

Table 2: Correspondence for transformations and transforming matrices in different inversions.

3 Group GT with its propositions

In this section, we will discuss a group, GT (General Transformation group), generated by U_1, V_1, W_1, U_2, V_2, and W_2. We will first present four observations. Then we will prove a theorem about this group's structure based on these observations in Section (3.1). Finally, we will discuss the center of this group in Section (3.2).

3.1 Structure of GT

Observation 1. $\det(U_1) = \det(V_1) = \det(W_1) = \det(U_2) = \det(V_2) = \det(W_2) = -1$. This implies that GT is a subgroup of $GL(4, \mathbb{Z}_{12})$.

Observation 2. Each of U_1, V_1, W_1, U_2, V_2, and W_2 has order 2.

Observation 3. Every 2-pair, AB, where $A, B \in \{U_1, V_1, W_1, U_2, V_2, W_2\}$ and $A \neq B$, has order 12. Moreover, any of these 2-pair commutes with another one.

Observation 4. Every 3-pair, ABC, where $A, B, C \in \{U_1, V_1, W_1, U_2, V_2, W_2\}$ and $A \neq B \neq C$, has order 2.

Convention 1. *For the interest of readability, to indicate the addition of a constant $n \in \mathbb{Z}_{12}$ to each component of $(a, b, c, d) \in {\mathbb{Z}_{12}}^4$, we write*

$$(a, b, c, d) + n := (a + n, b + n, c + n, d + n).$$

Lemma 3.1 (U_1-conjugation)**.** *For any $A \in \{U_1V_1, U_1W_1, U_1U_2\}$, there exists*

$$U_1 A^m {U_1}^{-1} = A^{-m}. \tag{3}$$

More generally, for any series $B = (U_1V_1)^{a_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3}$ where $a_i \in \mathbb{Z}_{12}$, there exists

$$U_1 B {U_1}^{-1} = B^{-1} \tag{4}$$

Proof. We first prove Equation (3) by showing an speacial case. Other case can be proved by the same process. Suppose $A = (U_1V_1)^m$, we then have

$$\begin{aligned} U_1 A^m {U_1}^{-1} &= U_1 (U_1V_1)^m {U_1}^{-1} \\ &= U_1U_1V_1(U_1V_1)^{m-1}{U_1}^{-1} \\ &= (V_1U_1)V_1(U_1V_1)^{m-2}{U_1}^{-1} \\ &\dots \\ &= (V_1U_1)^{m-1}V_1{U_1}^{-1} \\ &= (V_1U_1)^{m-1}V_1U_1 \\ &= (V_1U_1)^m \\ &= (U_1V_1)^{-m} = A^{-1}. \end{aligned}$$

Now we prove Equation (4) by using the result above. Suppose now that $A = (U_1V_1)^{a_1}$, $B = (U_1W_1)^{a_2}$, and $C = (U_1U_2)^{a_3}$, we have

$$\begin{aligned} U_1ABC(U_1)^{-1} &= U_1A{U_1}^{-1}U_1B{U_1}^{-1}U_1C{U_1}^{-1} \\ &= A^{-1}B^{-1}C^{-1} \\ &= (CBA)^{-1} = (ABC)^{-1}, \end{aligned}$$

(Noted that in the last line we use Observation (3)). Hence (4). In the following theorem we will provide a more general form of Equation (3). □

Theorem 3.2 (structure of GT)**.** *GT has following properties:*

(i) *The U_1-conjugate of any 2-pair element AB, where $A, B \in \{U_1, V_1, W_1, U_2, V_2, W_2\}$ and $A \neq B$ is its inverse.*

$$U_1(AB){U_1}^{-1} = (AB)^{-1}$$

(ii) (a) *Every element in GT can be uniquely expressed as*

$$U_1{}^k (U_1 V_1)^{a_1} (U_1 W_1)^{a_2} (U_1 U_2)^{a_3}, \tag{5}$$

where $k = 0, 1$, $a_i \in \mathbb{Z}_{12}$.

(b) *Hence GT has order* 432

(iii) *GT is isomorphic to $\mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12})$.*

Proof.

(i) For any 2-pair AB, where $A \neq B \neq U_1$, we have

$$(AB)U_1(AB)U_1 = ABU_1ABU_1 = (ABU_1)(ABU_1) = \text{Id},$$

following Observation (4). Hence,

$$U_1(AB)U_1 = (AB)^{-1}$$
$$U_1(AB)U_1{}^{-1} = (AB)^{-1},$$

following Observation (2). Otherwise, if one of A, B is U_1, without loss of generality, we suppose $A = U_1$, therefore,

$$U_1(U_1B)U_1{}^{-1} = U_1U_1BU_1{}^{-1} = BU_1{}^{-1} = BU_1 = (U_1B)^{-1},$$

here we use Observation (3).

(ii) (a) Because of Observation (3), every word in GT does not have two consecutive letters which are the same. We will first prove that every word can be expressed in form (5) by recurrence, then prove its uniqueness.

For every word with length 1, we prove it can be written in (5). Of course U_1 can, and for every other element A, we write it in $U_1U_1A = U_1(U_1A) = U_1(AU_1)^{11}$, following Observation (2) and (3). Specially, we observed that $V_2 = U_1(U_1V_1)(U_1U_2)^{11}$ and $W_2 = U_1(U_1V_1)(U_1W_1)^{11}$. For word with length 2, if it does not appear in (5), for instance, V_1W_1, we can rewrite it into

$$V_1U_1U_1W_1 = (V_1U_1)(U_1W_1) = (U_1V_1)^{11}(U_1W_1).$$

Others are the same. And specially, we notice that $W_2U_1 = (U_1V_1)^{11}(U_1W_1)$ and $V_2U_1 = (U_1V_1)^{11}(U_1U_2)$.

Now for uniqueness. We claim that for any $A, B \in \{U_1V_1, U_1W_1, U_1U_2\}$ and $A \neq B$, the intersection of the groups generated by A and B seperately is the identity,

$$\langle A \rangle \cap \langle B \rangle = \text{Id}. \tag{6}$$

This can be proved by multiplying A and B with a same vector, then compare vector form. Take U_1V_1 and U_1W_1 as an example:

$$(U_1V_1)^m(a,b,c,d)^T = (a,b,c,d)^T + m(a-b+c-d)$$
$$(U_1W_1)^n(a,b,c,d)^T = (a,b,c,d)^T + n(a-b)$$

(here we use convention (1)), the two vectors on the right hand side will be the same if and only if $m(a-b+c-d) = n(a-b)$ for $\forall(a,b,c,d) \in \mathbb{Z}_{12}{}^{\times 4}$, this implies $m = n =$

0. Hence (6). Furthermore, for any $A, B, C \in \{U_1V_1, U_1W_1, U_1U_2\}$ with $A \neq B \neq C$, the follwing relationship exists

$$\langle A\rangle \cap \langle B, C\rangle = \mathrm{Id}, \tag{7}$$
$$\langle U_1\rangle \cap \langle U_1V_1, U_1W_1, U_1U_2\rangle = \mathrm{Id}. \tag{8}$$

This can also be proved by the process above. We only prove (8) here. By writing these transformations into vector form, there is

$$(U_1V_1)^m(U_1W_1)^n(U_1U_2)^p(a,b,c,d)^T = (a,b,c,d) + m(a-b+c-d) + n(a-b) + p(-b+c)$$
$$U_1(a,b,c,d)^T = (-a+b+c, d, b-c+d, b).$$

The two vectors on the right hand side are the same if

$$\begin{cases} -a+b+c = (m+n+1)a-(m+n+p)b+(m+p)c-md \\ d = (m+n)a-(m+n+p-1)b+(m+p)c-md \\ b-c+d = (m+n)a-(m+n+p)b+(m+p+1)c-md \\ b = (m+n)a-(m+n+p)b+(m+p)c-(m-1)d \end{cases} \tag{9}$$

$$\begin{cases} a = (m+n+1)a-(m+n+p)b+(m+p)c-md \\ d = (m+n)a-(m+n+p-1)b+(m+p)c-md \\ c = (m+n)a-(m+n+p)b+(m+p+1)c-md \\ d = (m+n)a-(m+n+p)b+(m+p)c-(m-1)d \end{cases} \tag{10}$$

for $\forall(a,b,c,d) \in \mathbb{Z}_{12}{}^{\times 4}$. However, (9) has no solutions, while (10) has only trivial solution. Hence (8).

Now, suppose for some $a_i, b_i \in \mathbb{Z}_{12}$, $i = 1,2,3,4$ and $a_i \neq b_i$ for every i, that

$$V_1{}^m(U_1V_1)^{a_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3} = V_1{}^n(U_1V_1)^{b_1}(U_1W_1)^{b_2}(U_1U_2)^{b_3}.$$

If $m = n$, then

$$(U_1V_1)^{a_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3} = (U_1V_1)^{b_1}(U_1W_1)^{b_2}(U_1U_2)^{b_3}$$
$$(U_1V_1)^{a_1-b_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3} = (U_1W_1)^{b_2}(U_1U_2)_3^b$$
$$\cdots$$
$$(U_1V_1)^{a_1-b_1}(U_1W_1)^{a_2-b_2}(U_1U_2)^{a_3-b_3} = \mathrm{Id},$$

this implies $a_i = b_i$ for every $i = 1,2,3,4$ since (6) and (7)[4], which contradicts to our presuppose. If $n \neq m$, we have

$$V_1(U_1V_1)^{a_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3} = (U_1V_1)^{b_1}(U_1W_1)^{b_2}(U_1U_2)^{b_3}$$
$$V_1 = (U_1V_1)^{b_1-a_1}(U_1W_1)^{b_2-a_2}(U_1U_2)^{b_3-a_3}$$

This contradicts to (8). Hence (a).

(b) This is a direct result from (a).

$$432 = \frac{2 \times 12 \times 12 \times 12}{2^3}.$$

[4]Because the result implies that

$$(U_1V_1)^{a_1-b_1} = (U_1W_1)^{b_2-a_2}(U_1U_2)^{b_3-a_3}$$

(or other combinations), but this contradicts to (7).

(iii) Now we claim that $\langle U_1V_1, U_1W_1, U_1U_2\rangle$ is a normal subgroup of GT. For $\forall g \in GT$, by (5), $g = {U_1}^k (U_1V_1)^{a_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3}$, where $k = 0,1$ and $a_i \in \mathbb{Z}_{12}$. In this way,

$$g^{-1} = (U_1U_2)^{12-a_3}(U_1W_1)^{12-a_2}(U_1V_1)^{12-a_1}(U_1)^{-k}.$$

And for $\forall h \in \langle U_1V_1, U_1W_1, U_1U_2\rangle$, $h = (U_1V_1)^m(U_1W_1)^n(U_1U_2)^p$, where $(m,n,p) \in {\mathbb{Z}_{12}}^{\times 3}$.

If $k = 0$, then

$$ghg^{-1} = h$$

because of the communitivity mentioned in Observation (1). And if $k = 1$, then

$$ghg^{-1} = h^{-1},$$

which is obviously in $\langle U_1V_1, U_1W_1, U_1U_2\rangle$.

Finally, by (8), we have

$$GT = \langle U_1\rangle \ltimes \langle U_1V_1, U_1W_1, U_1U_2\rangle,$$

which is isomorphic to $\mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12})$ because of Observation (2) and (3).

□

Remark 1. *For completeness, we also note here that*

$$(U_1U_2)^p(a,b,c,d)^T = (a,b,c,d)^T + p(-b+c)$$

Remark 2. $(U_1V_1)^m$*,* $(U_1W_1)^n$*, and* $(U_1U_2)^p$ *can also be written in matrix form:*

$$(U_1V_1)^m = \begin{pmatrix} m+1 & -m & m & -m \\ m & -m+1 & m & -m \\ m & -m & m+1 & -m \\ m & -m & m & -m+1 \end{pmatrix}$$

$$(U_1W_1)^n = \begin{pmatrix} n+1 & -n & 0 & 0 \\ n & -n+1 & 0 & 0 \\ n & -n & 1 & 0 \\ n & -n & 0 & 1 \end{pmatrix} \tag{11}$$

$$(U_1U_2)^p = \begin{pmatrix} 1 & -p & p & 0 \\ 0 & -p+1 & p & 0 \\ 0 & -p & p+1 & 0 \\ 0 & -p & p & 1 \end{pmatrix}$$

This can be easily proved by induction.

Remark 3 (index of GT in $GL(4,\mathbb{Z}_{12})$)**.** *Generally, the order of* $GL(n,F)$ *where* F *is a field and* $|F| = q$ *is*

$$|GL(n,F)| = \prod_{k=0}^{n-1}\left(q^n - q^k\right). \tag{12}$$

Therefore,

$$|GL(4,\mathbb{Z}_{12})| = |GL(4,\mathbb{Z}_3)| \times |GL(4,\mathbb{Z}_4)|.$$

$|GL(4,\mathbb{Z}_3)| = (3^4-1)(3^4-3)(3^4-3^2)(3^4-3^3) = 24{,}261{,}120$ *according to (12), and*

$$|GL(4,\mathbb{Z}_4)| = 2^{4^2\cdot 2}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{2^4}\right) = 1{,}321{,}205{,}760$$

Recall Theorem 1 in ([6]) and Equation (2.3) in ([8]). Hence,

$$[GL(4,\mathbb{Z}_{12}) : GT] = \frac{24{,}261{,}120 \cdot 1{,}321{,}205{,}760}{3456} = 9{,}274{,}864{,}435{,}200.$$

3.2 The center of GT

Proposition 1. *The center of GT is a group contains all the elements in the following set*

$$\{(U_1V_1)^{6a_1}(U_1W_1)^{6a_2}(U_1U_2)^{6a_3} | a_i = 0, 1\}.$$

Proof. We first recall that the center of group GT is a subgroup $Z(GT)$ where for all $A \in Z(GT)$ and $B \in GT$, $AB = BA$. Now, consider a $A \in \langle U_1V_1, U_1W_1, U_1U_2\rangle$. A will be in $Z(GT)$ if it commutes with U_1. Thus,

$$U_1A = AU_1$$
$$U_1AU_1^{-1} = A = A^{-1},$$

which implies $A^2 = \mathrm{Id}$. Therefore, A has to be in the set above to have this property. On the other hand, if $A = U_1B$ for some $B = (U_1V_1)^{a_1}(U_1W_1)^{a_2}(U_1U_2)^{a_3}$ $(a_i \in \mathbb{Z}_{12})$,

$$U_1A = U_1U_1B = B$$

which does not necessarily equal to $AU_1 = U_1BU_1$ since U_1 and B do not commute. □

Remark 4. *By Proposition (1), we obtain the following:*

$$Z(GT) \cong C_2 \times C_2 \times C_2.$$

4 An important subgroup of GT

In this section, we will demonstrate a subgroup of GT, which has important musical significance. We name it as XNP group, which is generated by U_1, V_1 and W_1 and is a dihedral group of order 24.

4.1 XNP group and its properties

Observation 5. By repeatedly applying V_1 and W_1 on a dominant seventh chord or a half-diminished seventh chord in its root position, we can acquire a trajectory with all the 24 dominant sevenths and half-diminished sevenths. Below is an example of trajectory starting with a dominant seventh which has root C.

C^7 $E^{ø7}$ G^7 $B^{ø7}$ D^7 $F\sharp^{ø7}$ A^7 $C\sharp^{ø7}$ E^7 $A\flat^{ø7}$ B^7 $E\flat^{ø7}$ $G\flat^7$ $B\flat^{ø7}$ $D\flat^7$ $F^{ø7}$ $A\flat^7$ $C^{ø7}$ $E\flat^7$ $G^{ø7}$ $B\flat^7$ $D^{ø7}$ F^7 $A^{ø7}$ C^7

In this sequence, the letters denote the roots. Superscript 7 is used to denote a dominant seventh, while superscript ø7 is used to denote a hal-diminished seventh. This is the convention of marking the seventh chord.

This observation implies that XNP group contains at least 24 elements, because the 24 bijections V_1, W_1V_1, $V_1W_1V_1, \ldots$, $V_1(W_1V_1)^{11}$ are different.

Theorem 4.1. *From Observation (5), we obtain the following properties:*

(i) *U_1-conjugation*

$$U_1(V_1W_1)U_1^{-1} = (V_1W_1)^{-1}$$

(ii) *Every element in XNP can be uniquely expressed as*

$$(U_1V_1)^n(U_1W_1)^{12-n} \tag{13}$$

or

$$U_1(U_1V_1)^n(U_1W_1)^{13-n} \tag{14}$$

where $n \in \mathbb{Z}_{12}$.

(iii) *XNP is a dihedral group of order* 24.

Proof.

(i) We calculate the following

$$(V_1W_1)U_1(V_1W_1)U_1 = \mathrm{Id}$$
$$U_1(V_1W_1)U_1 = (V_1W_1)^{-1}$$
$$U_1(V_1W_1)U_1{}^{-1} = (V_1W_1)^{-1}$$

following Observation (2) and (4).

(ii) The expansion of the words in XNP is as same as in Theorem (3.2). We here present some of the expansionds, others could be obtained by mathematical induction.

$$V_1 = U_1U_1V_1$$
$$W_1V_1 = (W_1U_1)(U_1V_1) = (U_1V_1)(U_1W_1)^{11}$$
$$V_1W_1V_1 = U_1U_1V_1W_1U_1U_1V_1 = U_1(U_1V_1)(U_1W_1)^{11}(U_1V_1) = U_1(U_1V_1)^2(U_1W_1)^{11}.$$

(iii) We recall that a dihedral group of order n is a group generated by two element s and t, satisfying

$$s^n = \mathrm{Id} \qquad t^2 = \mathrm{Id} \qquad tst = s^{-1}.$$

In this case, we set $s = (V_1W_1)$ and $t = U_1$. Following Observation (2) and (3) and (i), we have

$$(V_1W_1)^{12} = \mathrm{Id} \qquad U_1{}^2 = \mathrm{Id} \qquad U_1(V_1W_1)U_1{}^{-1} = (V_1W_1)^{-1}.$$

And from Equation (13) and (14), we deduce that XNP has order 24. Hence (iii).

□

Proposition 2. *XNP is isomorphic to $\mathbb{Z}_2 \ltimes \mathbb{Z}_{12}$.*

Proof. For any $g \in \langle V_1W_1\rangle$ and $h \in$ XNP, we have

$$hgh^{-1} = (U_1)^k(V_1W_1)^p(V_1W_1)^m(V_1W_1)^{12-p}(U_1)^k$$
$$= (U_1)^k(V_1W_1)^m(U_1)^k.$$

This implies that $hgh^{-1} \in \langle V_1W_1\rangle$ for either $k = 0$ or $k = 1$. And we have

$$\langle U_1\rangle \cap \langle V_1W_1\rangle = \mathrm{Id},$$

hence XNP $= \langle U_1\rangle \ltimes \langle V_1W_1\rangle \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{12}$. □

Proposition 3. *By repeatedly applying U_1, V_1, and W_1 on any of the 24 dominant and half-diminished seventh chord, the chrod will be transformed back into itself after a trajectory of length 6. Below is an example of trajectory starting with C^7 (here we use convention mentioned in Proposition (5)).*

$$C^7 \xrightarrow{U_1(X)} E^{ø7} \xrightarrow{V_1(N)} G^7 \xrightarrow{W_1(P)} G^{ø7} \xrightarrow{U_1(X)} E\flat^7 \xrightarrow{V_1(N)} C^{ø7} \xrightarrow{W_1(P)} C^7$$

Because of Proposition (3), we can arrange all the seventh chords on a plane combining their relationships. The diagram is called a *Tonnetz*, after Euler's terminology and means *tone network* in German. XNP Tonnetz is shown in Figure (1). In this Tonnetz, every pink edge denotes a P transformation, bluish edge denotes a X transformation, and gray edge denotes a N transformation.

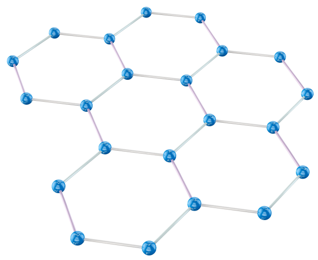

Figure 1: XNP Tonnetz. Every node in this Tonnetz is a seventh chord (either dominant seventh or half-diminished seventh).

4.2 Six transformations revisited

In this section, we will review our six transformations present in Section (2.2). We will first present an observation, and then present a structure of relationships between seventh chords.

Observation 6. Composite $W_1V_2W_2U_2 = \mathrm{Id}$. This can be proved by straightforward calculations.

This observation, combined with Proposition (3), enables us to construct a structure in space. In this structure, each layer is a XNP Tonnetz, and the connections between layers are the observation above. These four transformations together form a loop of transformations. We reshape it into a tetrahedron which makes conntections in space possible.

The following figure is the Tonnetz of dominant seventh and half-diminished seventh. Noted that each node in this diagram is a seventh chord, and each edge is a transformation.

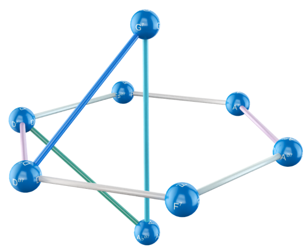

Figure 2: Basic unit of the Tonnetz of dominant seventh and half-diminished seventh. Note that the 4-loop which contains an edge of the hexagon and three edges in space is mentioned in Observation (6).

Figure 3: Tonnetz of dominant seventh and half-diminished seventh. For the interest of cleariity, two images of different angles of the Tonnetz are shown.

4.3 Case studies

Below we will present some example. These examples contain transformations mentioned above and has musical significance.

Example 1. In the first two measures of his first Symphony[5], Brahms showed an interplay from a dominant seventh to a half-diminished seventh, which completes a P transformation. This transformation creates abundant tension. Moreover, he used this transformation as a core motivation, which appears several times in other parts of his Symphony. The interplay can be expressed as

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 10 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ 6 \\ 3 \end{pmatrix}.$$

See Figure (4) for musical needs.

Figure 4: First two measures (partial) of Brahms' first Symphony. It is clear that the violin parts, viola part, and cello part together play a transformation from E♭^{7} to E♭^{ø7}.

Example 2. This example from *Parsifal*[6] makes full use of the harmonic function of the G transformation. After G-transforming a half-diminished seventh chord into a dominant seventh chord, he did not resolve it instantly, but replaced the target chord (B♭ major) with its parallel half-diminished seventh chord (B♭^{ø7}). Then he uses this operation several more times, realizing the root notes' constant fourth altering, which goes anticlockwise in the fifth clock, making the harmony constantly darken. This operation can be realized since that the root note's distance in a G transformation is the same as it is when a dominant seventh is resolved, and a half-diminished seventh chord is exactly a dominant seventh after a G transformation.

This series of transformations can be expressed in the form below:

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 9 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \\ 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 10 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 1 \\ 10 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 11 \\ 8 \end{pmatrix}.$$

See Figure (5) for musical needs.

[5]Symphony No.1 in C minor, Op.68.

[6]Parsifal, WWV 111. Act I, *Mein Sohn Amfortas, Bist du Am Amt?, Wehe! Wehe mir der Qual!*

Figure 5: Amfortas' *Agony* Aria from Act I of *Parsifal*, mm. 1369-72

5 Conclusions and a conjecture

In this paper, we constructed a group $U_1 \ltimes \langle U_1 V_1, U_1 W_1, U_1 U_2 \rangle$, namely GL. We showed that it is a group of order 3456 and every element in this group can be uniquely written in the form of (5). We furthermore focused on a subgroup of GL, XNP, and found some musical properties in this group.

Base on the goal to transform a chord smoothly into one another, we can obtain the following proposition.

Proposition 4. *Every n-chord, that is, a chord contains n pitch classes has at least*

$$\sum_{k=1}^{n-1} k = (n-1) + (n-2) + \cdots + 1$$

transformations that preserve two or more than two pitch classes.

Proof. Every n-chord can be expressed as a n-gon on the musical clock. A symmetry about the midperpendicular of side or diagonal of this n-gon will at least preserve two pitch classes. By induction, there are $\sum_{k=1}^{n-1} k = (n-1)+(n-2)+\cdots+1$ midperpendicular possible for a n-gon. Hence Proposition (4). □

By comparing the result of Fiore in ([5]) and in this paper, we have the following conjecture.

Conjecture 1. *All of the* $\sum_{k=1}^{n-1} k = (n-1)+(n-2)+\cdots+1$ *tranformations of a n-chord,* $\{U_1, V_1, V_2, \cdots, V_{\lceil \frac{n(n-1)}{2} \rceil -1}\}$*, can generate a group. Every element in this group can be uniquely written as*

$$U_1{}^k (U_1 V_1)^{a_1} (U_1 V_2)^{a_2} \cdots (U_1 V_{\lceil \frac{n(n-1)}{2} \rceil})^{a_{\lceil \frac{n(n-1)}{2} \rceil}}, \tag{15}$$

where $k = 0, 1$ *and* $a_i \in \mathbb{Z}_{12}$.

And this group is isomorphic to $\mathbb{Z}_2 \ltimes \mathbb{Z}_{12}{}^{\times (\lceil \frac{n(n-1)}{2} \rceil)}$.

Further studies can focus on this conjecture. Although chords with more than 4 pitch classes rarely appears in music, it is better to understand transformations between chords in a more general way. This will provide us a bigger picture and will give us better understandings to the present theories.

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