Representations of $\mathfrak{sl}(2, \mathbb{F})$

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Abstract

In this report, we will give a short review of the concepts related to linear Lie algebra and tensor products, construct irreducible representations of $\mathfrak{sl}(2, \mathbb{F})$ with arbitrary positive integer dimension, present a proof of Clebsch-Gordan formula, and provide formulas for decomposing symmetric tensor space and skew-symmetric tensor space over an irreducible representation into other irreducible representations.

1 Introduction

Following Exercise 4 of [Hum72], we prove the following theorem regarding the representations of $\mathfrak{sl}(2, \mathbb{F})$, where $\mathbb{F}$ is a algebraically closed field with characteristic 0 (throughout the paper).

Theorem 1. *For every nonnegative integer m, there exists a unique irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ of dimension $m + 1$, up to equivalence.*

We denote the unique irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ with dimension m by V_m in the following text. Naturally, we think about how new representations can be constructed using V_i. It turns out that the tensor product of two such representations can be decomposed into irreducible representations with lower dimensions, according to the following formula:

Theorem 2. *Let V_m and V_n be irreducible representations of $\mathfrak{sl}(2, \mathbb{F})$ with dimensions m and n ($m \geq n$) respectively. We have*

$$V_m \otimes V_n \cong \bigoplus_{i=0}^{n} V_{m+n-2i}.$$

This formula is often referred to as Clebsch-Gordan formula.

Finally, with the fact that $V_m \otimes V_m \cong \operatorname{Sym}^2 V_m \oplus \Lambda^2 V_m$ (which we will prove in Proposition 4), we can deduce a formula for decomposing $\operatorname{Sym}^2 V_m$ and $\Lambda^2 V_m$ respectively, as follows:

Proposition 1. *Let V_m be an irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$. We have*

$$\operatorname{Sym}^2 V_m \cong \bigoplus_{i=0}^{\lfloor \frac{m}{2} \rfloor} V_{2m-4i}$$

and

$$\Lambda^2 V_m \cong \bigoplus_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} V_{2m-2-4i}.$$

2 Preliminaries

In this section, we will give the basic definitions relating to this topic. Let V be a finite dimensional vector space over the field $\mathbb{F}$, we start by giving the definition of $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$.

Definition 1 (General linear algebra)**.** Let V be a finite dimensional vector space. The general linear algebra is the set of all endomorphisms of V, End V, with the bracket defined as $[x,\ y] = xy - yx$ for all $x, y \in \text{End } V$, denoted by $\mathfrak{gl}(V)$.

Definition 2 (Special linear algebra)**.** Let V be a vector space over F, dim $V = n$. The special linear algebra is all traceless endomorphism over V, denoted by $\mathfrak{sl}(V)$ or $\mathfrak{sl}(n, \mathbb{F})$. The bracket is defined as same as for $\mathfrak{gl}(V)$.

Example 1. When $n = 2$, $\mathfrak{sl}(2, \mathbb{F})$ will be the set of 2×2 traceless matrices. The following is a basis of $\mathfrak{sl}(2, \mathbb{F})$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is called the standard basis of $\mathfrak{sl}(2, \mathbb{F})$, and the following relations are worthnoting:

$$[h,\ x] = 2x \qquad [h,\ y] = -2y.$$

Remark 1. Note that $\text{tr}\,([x,\ y]) = \text{tr}\,(xy - yx) = \text{tr}\,(xy) - \text{tr}\,(yx) = 0$. This implies $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$.

Now we give the definition of a representation of a Lie algebra L along with the definition of an L-module.

Definition 3 (Representation)**.** Let L be a Lie algebra. A representation of L is a homomorphism $\phi : L \to \mathfrak{gl}(V)$.

Definition 4 (L-module)**.** Let L be a Lie algebra. Let V be a vector space, endowed with an operation $L \times V \to V$ (denoted $(x,\ v) \mapsto x.v$ or just xv), is called an L-module if the following conditions are satisfied:

(M1) $(ax + by).v = a(x.v) + b(y.v)$,

(M2) $x.(av + bw) = a(x.v) + b(x.w)$,

(M3) $[x,\ y].v = xy.v - yx.v$. $(x, y \in L, v, w \in V, a, b \in \mathbb{F})$.

Remark 2. If $\phi : L \to \mathfrak{gl}(V)$ is a representation, then V can be viewed as an L-module via the action $x.v = \phi(x)(v)$, which obviously satisfies the definitions. On the other hand, if V is an L-module, then $\phi : L \to \mathfrak{gl}(V)$ will be a representation.

Definition 5 (L-invariant subspace)**.** Let $\phi : L \to \mathfrak{gl}(V)$ be a representation. A subspace W of V is L-invariant if, for all $x \in L$ and $w \in W$, one has $\phi_x w \in W$.

Definition 6 (Completely reducible)**.** Let L be a Lie algebra. A representation $\phi : L \to \mathfrak{gl}(V)$ is said to be completely irreducible if $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ where the V_i are L-invariant subspaces.

The following theorem by Weyl is the foundation to our later discussions, as $\mathfrak{sl}(V)$ is semisimple.

Theorem 3 (Weyl)**.** *Let L be a semisimple Lie algebra and $\phi : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of L. Then ϕ is completely reducible.*

We will mainly use L-modules in the following discussions.

Definition 7 (Semisimple operator)**.** Call $x \in \text{End } V$ (V finite dimensional) semisimple if the roots of its minimal polynomial over $\mathbb{F}$ are all distinct.

Lemma 1. The zeros (over $\mathbb{F}$) of the minimal polynomial of an operator T are the eigenvalues of T.

Proof. Let p be the minimal polynomial of T and $\lambda \in \mathbb{F}$ be a root of p. Thus we can write p as $p(z) = (z - \lambda)q(z)$ where q is a monic polynomial. Now, $0 = p(T) = (T - \lambda I)q(T)$. Thus

$$((T - \lambda I)q(z))u = 0$$

for all $u \in V$. Since deg $q =$ deg $p - 1$, there exists at least one $u \in V$ such that $q(T)u \neq 0$. Hence we can conclude λ is an eigenvalue of T.

Now suppose λ is an eigenvalue of T. Thus we can write p as $p(T) = (T - \lambda I)q(T) = 0$. This would imply $p(z) = (z - \lambda)q(z) = 0$, and λ is a root of p. □

Proposition 2 (Jordan-Chevalley decomposition)**.** *Let V be a finite dimensional vector space over $\mathbb{F}$, $x \in \text{End } V$. Then there exist unique $x_s, x_n \in \text{End } V$ satisfying the conditions: $x = x_s + x_n$, x_s semisimple, x_n nilpotent, x_s and x_n commute.*

Since the map $L \to \text{ad } L$ is a bijection, each $x \in L$ determines unique $s, n \in L$ such that $\text{ad } x = \text{ad } s + \text{ad } n$. This is called the abstract Jordan-Chevalley decomposition.

Definition 8 (Toral subalgebra)**.** For a Lie algebra L, the set of elements $x \in L$ with nonzero semisimple part x_s in the abstract Jordan-Chevalley decomposition is called a toral subalgebra of L.

Proposition 3. *A toral subalgebra of L is abelian.*

Proof. Let T be toral. Now we aim to show $\text{ad}_T\, x = 0$ for all $x \in T$. Since ad x is diagonalizable (ad x semisimple and $\mathbb{F}$ algebraically closed), we only have to show ad x has no nonzero eigenvalues. Now suppose ad x has nonzero eigenvalue, and there exists nonzero $y \in T$ such that ad $x(y) = ay$ ($a \neq 0$). By the definition of Lie algebra, we have ad $y(x) = -ay$. Note that ad $y(-ay) = [y,\ -ay] = -a[y,\ y] = 0$. This implies ad $y(x) = -ay$ is in the 0-eigenspace of ad y. On the other hand, since $y \in T$ is semisimple, we can decompose x into an eigenbasis of ad y

$$x = x_0 + \sum x_{\lambda_i}$$

where x_0 belongs to the 0-eigenspace. Now apply ad y to x, we have

$$\text{ad } y(x) = \text{ad } y(x_0) + \sum \text{ad } y(x_{\lambda_i}) = \sum_{\lambda \neq 0} \lambda_i x_{\lambda_i}.$$

This implies ad $y(x)$ is made up completely with eigenvectors from nonzero eigenspaces. However, from above, we have already argued that ad $y(x)$ is in the 0-eigenspace of ad y, and we have a contradiction.

Thus, ad $x(y) = 0$ for all $x, y \in T$. Hence the result. □

And we finally come to the definitions of symmetric and skew-symmetric tensor product. In this case, we will follow the definition in [Rom08]. Let S_p be the symmetric group on $\{1, \dots, p\}$. For each $\sigma \in S_p$, the multilinear map $f_\sigma : V^p \to V^{\otimes p}$

$$f_\sigma(x_1, \dots, x_p) = x_{\sigma 1} \otimes \cdots \otimes x_{\sigma p}$$

determines a unique linear operator λ_σ on $V^{\otimes p}$ for which

$$\lambda_\sigma(x_1 \otimes \cdots \otimes x_p) = x_{\sigma 1} \otimes \cdots \otimes x_{\sigma p}.$$

And now we are ready to give the definitions.

Definition 9 (Symmetric and skew-symmetric tensor product)**.** Let V be a finite dimensional vector space.

1. A tensor $t \in V^{\otimes p}$ is symmetric if $\lambda_\sigma t = t$ for all $\sigma \in S_p$. The set $\mathrm{Sym}^p\ V = \{t \in V^{\otimes p} : \lambda_\sigma t = t \text{ for all } \sigma \in S_p\}$ is a subsapce of $V^{\otimes p}$, called the symmetric tensor space of degree p over V.

2. A tensor $t \in V^{\otimes p}$ is skew-symmetric if $\lambda_\sigma t = (-1)^\sigma t$ for all $\sigma \in S_p$. The set $\Lambda^p\ V = \{t \in V^{\otimes p} : \lambda_\sigma t = (-1)^\sigma t \text{ for all } \sigma \in S_p\}$ is called the skew-symmetric (or antisymmetric) tensor space of degree p over V.

The following proposition will be useful later.

Proposition 4. *Let V be a finite dimensional vector space. We have*

$$V \otimes V \cong \mathrm{Sym}^2\ V \oplus \Lambda^2\ V.$$

Proof. Let $v_1, \dots, v_n$ be a basis of V, then $\{v_i \otimes v_j : 1 \leq i, j \leq n\}$ will be a basis of $V \otimes V$. For each $v_i \otimes v_j$, consider the identity

$$v_i \otimes v_j = \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i) + \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i).$$

Note that $v_i \otimes v_j + v_j \otimes v_i \in \mathrm{Sym}^2\ V$ by the following argument. When $p = 2$, $S_p \cong \mathbb{Z}/2\mathbb{Z}$, the only permutations are the identity and $(1\ 2)$. The case $\lambda_{\mathrm{id}}(v_i \otimes v_j + v_j \otimes v_i)$ is trivial, and $\lambda_{(1\ 2)}(v_i \otimes v_j + v_j \otimes v_i) = v_j \otimes v_i + v_i \otimes v_j = v_i \otimes v_j + v_j \otimes v_i$. Thus the result by definition. Also note that $v_i \otimes v_j - v_j \otimes v_i \in \Lambda^2\ V$ by a similar argument. When $\sigma = (1\ 2)$, $(-1)^\sigma = -1$; thus $\lambda_{(1\ 2)}(v_i \otimes v_j - v_j \otimes v_i) = v_j \otimes v_i - v_i \otimes v_j = (-1)^\sigma (v_i \otimes v_j - v_j \otimes v_i)$.

Now we show that $\mathrm{Sym}^2\ V \cap \Lambda^2\ V = 0$. This is obvious, as $v_i \otimes v_j + v_j \otimes v_i = v_i \otimes v_j - v_j \otimes v_i$ implies $v_j \otimes v_i = 0$, which further implies $v_i \otimes v_j = 0$. Thus the result, as desired. □

And now we are ready to begin constructing irreducible representations of $\mathfrak{sl}(2, \mathbb{F})$.

3 Irreducible representations

Since any representation of $\mathfrak{sl}(2, \mathbb{F})$ can be decomposed into irreducible representations, we can gain better understandings of the representations by looking at the irreducible representations. In this section, we will construct irreducible representations of any arbitrary dimension, following Exercise 4 in Chapter 7 of [Hum72].

In the following text of this section, we will use L to denote $\mathfrak{sl}(2, \mathbb{F})$, and we will use x, y, h as same as in Example 1.

First, we will take a look of ad h. Making use of the relations presented in Example 1, we can conclude that

$$\text{ad } h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and note that this matrix has three distinct eigenvectors: 0, 2, -2. Thus h is semisimple by definition. Also by direct computation, we have $[x,\ h] = -2$, $[x,\ y] = 0$, and $[y,\ h] = 2$. Given this, we also have

$$\text{ad } x = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{ad } y = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The primary goal of this section is to prove Theorem 1. The following definitions and propositions will be useful in our proof.

Definition 10 (Weight, weight spaces)**.** Let h be defined as in Example 1, V be an arbitrary L-module. Since h semisimple, V can be decomposed into direct sum of eigenspaces (following Lemma 1): $V_\lambda = \{v \in V : h.v = \lambda v\}$, $\lambda \in \mathbb{F}$. Whenever $V_\lambda \neq 0$, we call λ a weight of h in V and V_λ a weight space.

We will see in the following lemma how x and y (all as defined in Example 1) moves a certain $v \in V$ between weight spaces.

Lemma 2. If $v \in V$, then $x.v \in V_{\lambda+2}$, $y.v \in V_{\lambda-2}$.

Proof. We have $h.(x.v) = [h,\ x].v + x.(h.v) = 2x.v + x(\lambda v) = (\lambda + 2)x.v$. For $y.v$ the proof is the same. □

Since V finite dimensional and is a direct sum of its weight spaces, there exists an weight space $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. For such V_λ, any nonzero $v \in V_\lambda$ is called a maximal vector.

The following lemma from [Hum72] is important.

Lemma 3. Assume V is an irreducible L-module. Let $v_0 \in V_\lambda$ be a maximal vector; set $v_{-1} = 0$, $v_i = (1/i!)y^i.v_0$. We have

(a) $h.v_i = (\lambda - 2i)v_i$,

(b) $y.v_i = (i + 1)v_{i+1}$,

(c) $x.v_i = (\lambda - i + 1)v_{i-1}$.

We take a closer look of this lemma. (a) implies each v_i is linearly independent, since they are in different eigenspaces of h. Let m be the smallest integer such that $v_m \neq 0$ and $v_{m+1} = 0$. This implies the subspace with basis $v_0, v_1, \ldots, v_m$ is a submodule of V. However, since V is irreducible, this submodule must be V itself. In addition, when $i = m + 1$, (c) implies $x.v_{m+1} = 0 = (\lambda - m)v_m$. Since $v_m \neq 0$, we have $\lambda = m$. Considering the two observations, we can conclude that the weight of a maximal vector is an integer.

In the next proposition, we will demonstrate a way to construct irreducible representations of arbitrary dimension.

Proposition 5. *Let $\mathcal{R} = \mathbb{F}[X, Y]$ be the polynomial algebra in two variables. The action of L on $\mathcal{R}$ is defined by the derivation rule: $z.fg = (z.f)g + f(z.g)$, for $z \in L$, $f, g \in \mathcal{R}$. This action is well-defined and $\mathcal{R}$ becomes a L-module. Moreover, the subspace of homogeneous polynomials of degree m, with basis X^m, $X^{m-1}Y$, $\ldots$, Y^m, is invariant under L and irreducible of highest weight m.*

Remark 3. We have the following relation:

$$x.X = 0 \qquad x.Y = X \qquad y.X = Y \qquad y.Y = 0 \qquad h.X = X \qquad h.Y = -Y.$$

And we propose a lemma to prove this proposition.

Lemma 4. When the basis of L acts on elements of the form X^kY^{m-k} $(k \neq 0, m)$, we have:

$$\begin{aligned} x.X^kY^{m-k} &= kX^{k+1}Y^{m-k-1} \\ y.X^kY^{m-k} &= kX^{k-1}Y^{m-k+1} \\ h.X^kY^{m-k} &= kX^kY^{m-k} - (m-k)X^kY^{m-k}. \end{aligned}$$

Proof. We prove the first formula by induction on m. In the case when $m = 2$, $k = 1$, with direct computation we have $x.XY = X^2$. Assume the formula holds for $m = p$ and arbitary k satisfying the condition. For $m = p + 1$, let k be arbitary and satisfies the condition,

$$\begin{aligned} x.X^kY^{p+1-k} &= (x.X^kY^{p-k})Y + X^kY^{p-k}(x.Y) \\ &= kX^{k+1}Y^{p-k-1}Y + X^kY^{p-k}X \\ &= kX^{k+1}Y^{p-k} + X^{k+1}Y^{p-k} \\ &= (k+1)X^{k+1}Y^{(p+1)-k-1}. \end{aligned}$$

Thus the formula. The second formula can be proved in a similar fashion. For the third formula, we investigate the following formula first: $h.X^k = kX^k$, $h.Y^k = -kY^k$. We prove these formulas by induction on k. When $k = 1$, by direct computation, $h.X = X$. Assume when $k = p$ the formula holds, consider the case when $k = p + 1$: $h.X^{p+1} = (h.X^p)X + X^p(x.X) = pX^pX + X^pX = (p+1)X^{p+1}$. The second formula can be proved similarly. Hence, $h.X^kY^{m-k} = (h.X^k)Y^{m-k} + x^k(h.Y^{m-k}) = kX^kY^{m-k} - (m-k)X^kY^{m-k}$.

As desired. □

Remark 4. Specifically, when $k = m$, $x.X^kY^{m-k} = x.X^m = 0$; when $k = 0$, $y.X^kY^{m-k} = y.Y^m = 0$. This follows Remark 3.

Now we are ready to prove Proposition 5.

Proof of Proposition 5. We first prove this action is well defined. $(z + z').fg = z.fg + z'.fg$ and $(cz).fg = cz.fg$ (c a scalar) since $z, z' \in L$ are linear. For the preservation of Lie bracket, we have

$$\begin{aligned} [z,\ z'].fg = (zz' - z'z).fg &= zz'.fg - z'z.fg \\ &= z.((z'.f)g + f(z'.g)) - z'((z.f)g + f(z.g)) \\ &= (zz'.f)g + (z'.f)(z.g) + (z.f)(z'.g) + f(zz'.g) \\ &\quad - (z'z.f)g - (z.f)(z'.g) - (z'.f)(z.g) - f(z'z.g) \\ &= [z.fg,\ z'.fg]. \end{aligned}$$

Preservation of Lie bracket verifies (M3) in Definition 4. Thus $\mathcal{R}$ is a L-module.

Let V be a subspace of $\mathcal{R}$ spanned by X^m, $X^{m-1}Y$, ..., Y^m. Obviously V is invariant under L, following Lemma 4. For the irreducibility of V, we prove for every W a proper subspace of V, W is not invariant under L. Assume V is spanned by $\{\sum_{k=0}^m c_{j,k}X^kY^{m-k} : 0 \leq j \leq m, c_{j,k} \in \mathbb{F}\}$, where for some j, every $c_{j,k} = 0$ (since W is properly contained in V). Now, for some fixed j_0, let k_0 be the smallest index where $c_{j_0,k} \neq 0$. We have $x^{m-k_0}. \sum_{k=0}^m c_{j_0,k}X^kY^{m-k} = X^m$ following Lemma 4 and Remark 4. By applying the second formula in Lemma 4 repeatedly, we have X^m, $X^{m-1}Y$, ..., Y^m are all in W since W is invariant under L. However, this contradicts to our initial assumption that W is properly contained in V. Thus V is irreducible, as desired.

Finally, following the third formula of Lemma 4, the weights are m, $m-2$, ..., $-m$, where m is the heighest weight. □

Corollary 1. Each weight space of the representation has dimension 1.

Proof. Indeed. There are $m+1$ weight spaces, and V (as defined in the proof) has dimension $m+1$. □

Remark 5. This corollary implies that if a representation has a weight with multiplicity greater than 1, then it can be reduced into smaller representations.

The proposition provides us with a way to prove the exsitence of irreucible representation of arbitrary dimension, and now we are ready to prove Theorem 1.

Theorem 1. *For every nonnegative integer m, there exists a unique irreducible representation of $\mathfrak{sl}(2,\mathbb{F})$ of dimension $m+1$, up to equivalence.*

Proof. Denote $V_m \subset \mathbb{F}[X,Y]$ as a subspace spanned by X^m, $X^{m-1}Y$, ..., Y^m. When $m=0$, $\phi : \mathfrak{sl}(2,\mathbb{F}) \to \mathfrak{gl}(V_0) \cong I$ is a representation of dimension $1 = 0+1$. For general m, the representation $\phi : \mathfrak{sl}(2,\mathbb{F}) \to \mathfrak{gl}(V_m)$ will do, following Proposition 5.

Suppose ϕ and ψ are two different irreducible representations of $\mathfrak{sl}(2,\mathbb{F})$ of dimension 2, with corresponding L-module V and W. Since $\mathfrak{sl}(2,\mathbb{F})$ is semisimple and the representations are irreducible, we aim to prove they share the same highest weight. Assume ϕ has maximal vector $v_0 \in V_\lambda$, use the same classification as in Lemma 3, with $v_i = (1/i!)y^i.v_0$, we have $x.v_i = (\lambda - i + 1)v_{i-1}$. Since both V and W has m weight spaces, this forces $\lambda = m$. Thus ϕ and ψ are equivalent. □

Due to uniqueness, we denote the irreducible representation with highest weight m by V_m, without any confusion.

4 Decomposing tensor product

As mentioned in [Hum72], one way to "fabricate new L-modules from old ones" is to take tensor product. Let V and W be two L-modules of a Lie algebra L. The operation of L on $V \otimes W$ is defined using Leibniz's rule: $x.(v \otimes w) = (x.v) \otimes w + v \otimes (x.w)$, $(x \in L,\ v \in V,\ w \in W)$. We will show $V \otimes W$ is indeed an L-module. (M1) and (M2) in Definition 4 holds because of the propositions of tensor product. Now we check (M3):

$$\begin{aligned}
[x,\ y].(v \otimes w) &= ([x,\ y].v) \otimes w + v \otimes ([x,\ y].w) \\
&= (xy.v - yx.v) \otimes w + v \otimes (xy.w - yx.w) \\
&= ((xy.v) \otimes w + v \otimes (xy.w)) - ((yx.v) \otimes w + v \otimes (yx.w)) \\
&= xy.(v \otimes w) + yx.(v \otimes w).
\end{aligned}$$

Recall Theorem 3, it is natural to ask how can we decompose the tensor product of two irreducible representations V_m and V_n.

The following observation will be useful later: suppose $v \in V$ has weight λ_1, $w \in W$ has weight λ_2, then $v \otimes w \in V \otimes W$ has weight $\lambda_1 + \lambda_2$. The proof is straightforward. $h.(v \otimes w) = (h.v) \otimes w + v \otimes (h.w) = \lambda_1 v \otimes w + v \otimes \lambda_2 w = (\lambda_1 + \lambda_2)(v \otimes w)$.

Now we can proof Theorem 2 from the introdcution.

Theorem 2. *Let V_m and V_n be irreducible representations of $\mathfrak{sl}(2,\mathbb{F})$ with dimensions m and n $(m \geq n)$ respectively. We have*

$$V_m \otimes V_n \cong \bigoplus_{i=0}^{n} V_{m+n-2i}.$$

Proof. The highest weight of $V_m \otimes V_n$ will be $m+n$ by definition. By the proof of Proposition 5, V_m has weights $m-2r$ $(0 \leq r \leq m)$, V_n has weights $n-2s$ $(0 \leq s \leq n)$. Thus weights of $V_m \otimes V_n$ will be in the form $m+n-2(r+s)$ following the observation. Now we consider the multiplicity of $m+n-2(r+s)$. Let $p=r+s$ in the following proof for conciseness.

If $0 \leq p \leq n$ (note that we assumed $n \leq m$), we have $s=p-r$. Since r ranges from 0 to m, we have $0 \leq s \leq p$, giving $p+1$ options for s. Thus the multiplicity will be $p+1$ when $0 \leq p \leq n$.

If $n+1 \leq p \leq m-1$, we have $r=p-s$. Since $0 \leq s \leq n$ and $p > n$, $p-n \leq r \leq p$, giving $n+1$ options for r. Thus the multiplicity will be $n+1$ when $n+1 \leq p \leq m$.

If $m \leq p \leq m+n$, similarly, $r=p-s$. By the range of s, we have $p-n \leq r \leq p$. However, since $p > m$, also consider the range of r, we have $p-n \leq r \leq m$, giving $m+n+1-p$ options for r. In this case, we can denote p as $m+n-t$ $(0 \leq t \leq n)$. Thus we can conclude that the weight $m+n-2p=-(m+n)+2t$ has multiplicity $m+n+1-p=t+1$.

Now we can conclude that the maximal multiplicity is reached when the weight is $m-n$, and when $0 \leq i \leq n$, the multiplicities of $m+n-2i$ and $-(m+n)+2i$ are the same.

To conclude the formula, we only have to consider that the difference between the multiplicities of two consecutive nonnegative weights, which is 1 for the difference between $m+n-2i$ and $m+n-2(i-1)$ when $1 \leq i \leq n$, and 0 otherwise. Hence the result. $\square$

We will illustrate this result with an example.

Example 2. Consider $V_2 \otimes V_1$. We denote the basis of V_2 as v_0, v_1, v_2 with weight $2, 0, -2$ respectively, denote the basis of V_1 as w_0, w_1 with weight $1, -1$ respectively. For conciseness, we use $\lambda(u)$ to denote the weight of a vector u. Now we compute a basis of $V_2 \otimes V_1$ and the corresponding weights:

$$\begin{aligned}
\lambda(v_0 \otimes w_0) &= \lambda(v_0) + \lambda(w_0) = 3 & \lambda(v_0 \otimes w_1) &= \lambda(v_0) + \lambda(w_1) = 1 \\
\lambda(v_1 \otimes w_0) &= \lambda(v_1) + \lambda(w_0) = 1 & \lambda(v_1 \otimes w_1) &= \lambda(v_1) + \lambda(w_1) = -1 \\
\lambda(v_2 \otimes w_0) &= \lambda(v_2) + \lambda(w_0) = -1 & \lambda(v_2 \otimes w_1) &= \lambda(v_2) + \lambda(w_1) = -3.
\end{aligned}$$

Thus, $V_2 \otimes V_1$ has weights $3, 1, -1, -3$, and the multiplicities of 1 and -1 are 2. Note that $3, 1, -1, -3$ are all the weights of V_3, and $1, -1$ are all the weights of V_1. Hence $V_2 \otimes V_1 \cong V_3 \oplus V_1$. This agrees with the above theorem.

Recall Proposition 4, since we already have a formula for $V_m \otimes V_m$, what will be the decomposition for $\mathrm{Sym}^2\, V_m$ and $\Lambda^2\, V_m$? We only need to find one of the formulas, and the other can be derived by removing corresponding terms. We will start by considering $\Lambda^2\, V_m$. The following theorem from [Rom08] will be useful in our discussion.

Theorem 4. *Let V be a finite dimensional vector space over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. For $p \leq n$, the skew-symmetric tensor space $\Lambda^p\, V$ is isomorphic to the algebra $\mathbb{F}_p^-[e_1, \ldots, e_n]$ of anticommutative homogeneous polynomials of degree p, via the isomorphism*

$$\tau\left(\sum \alpha_{i_1,\ldots,i_p} e_{i_1} \otimes \cdots \otimes e_{i_p}\right) = \sum \alpha_{i_1,\ldots,i_p} \left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right).$$

Instead of considering tensor products explicitly, this theorem allows us to express the basis using the wedge product, which has similar algebraic properties as the tensor product. Suppse the a basis of V_m is $v_0, \ldots, v_m$, then the basis of $\Lambda^2\, V_m$ will be $\{v_i \wedge v_j : 0 \leq i < j \leq m\}$. Similar to tensor product, if v has weight λ_1 and u has weight λ_2, $v \wedge u$ will have weight $\lambda_1 + \lambda_2$.

Now we are confident to proof Proposition 1, following a similar fashion as the proof of Theorem 2.

Proposition 1. *Let V_m be an irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$. We have*

$$\mathrm{Sym}^2\ V_m \cong \bigoplus_{i=0}^{\lfloor \frac{m}{2} \rfloor} V_{2m-4i}$$

and

$$\Lambda^2\ V_m \cong \bigoplus_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} V_{2m-2-4i}.$$

Proof. Firstly, we prove the second formula. Note by the definition of wedge product, the highest weight of $\Lambda^2\ V_m$ will be $m + (m-2) = 2m-2$, and the weights are in the form $(2m-2) - 2p$ where $0 \leq p \leq 2m-2$. Similar to the proof of Theorem 2, let $p = r + s$, where $0 \leq r \leq s \leq m$ (note that s and r can be the same since the corresponding weights are $2m-2r$ and $2m-2-2s$).

If $0 \leq p \leq m-1$, we have $s = p - r \geq r$. Thus $2r \leq p$, $0 \leq r \leq p/2$. On the other hand, since the r is an integer, the maximal of r will be $\lfloor p/2 \rfloor$. This give $\lfloor p/2 \rfloor + 1$ options for r. Hence, in this case, the multiplicity of $2m-2-2p$ will be $\lfloor p/2 \rfloor + 1$. Note that if m is even, the maximal multiplicity will be $\lfloor (m-2)/2 \rfloor = m/2 - 1$, when $p = m-2$. If m is odd, the maximal multiplicity will be $\lfloor (m-1)/2 \rfloor = \lfloor (m-1)/2 \rfloor - 1$ when $p = m-1$.

If $m \leq p \leq 2m-2$, we have $r = p - s \leq s$. Thus $2s \geq p$, $m \geq s \geq p/2$. This leaves m with $m - \lfloor p/2 \rfloor$ options. We can rewrite $p = 2m-2-t$ where $0 \leq t \leq m-2$. We multiplicity will change to $m - \lfloor (2m-2-t)/2 \rfloor = m - (-\lfloor t/2 \rfloor - m - 1) = \lfloor t/2 \rfloor + 1$.

This leads to the conclusion that for $0 \leq i \leq m-2$, the multiplicities of $2m-2-2i$ and $-2m+2+2i$ are the same.

Now we consider the difference between the multiplicities of two consecutive weights: $2m-2-2p$ and $2m-2-2(p-1)$ where $1 \leq p \leq m-1$. If p is even, then $\lfloor p/2 \rfloor + 1 = (\lfloor (p-1)/2 \rfloor + 1) + 1$; if p is odd, then $\lfloor p/2 \rfloor + 1 = \lfloor (p-1)/2 \rfloor + 1$. Thus, the final sum will contain terms when p is even, which further implies $2p$ is a multiply of 4. Thus the result.

By removing the corresponding terms from the decomposition of $V_m \otimes V_m$, we have the first equation. $\square$

This will be more clear with an example.

Example 3. Consider $V_3 \otimes V_3$. Denote a basis of V_3 by v_0, v_1, v_2, v_3 with weight $3, 1, -1, -3$ respectively.

By Theorem 2, we have $V_3 \otimes V_3 \cong V_6 \oplus V_4 \oplus V_2 \oplus V_0$. Now compute a basis of $\Lambda^2\ V_3$. Still, we use $\lambda(u)$ to denote the weight of a vector u.

$$\begin{aligned} \lambda(v_0 \wedge v_1) &= \lambda(v_0) + \lambda(v_1) = 4 & \lambda(v_0 \wedge v_2) &= \lambda(v_0) + \lambda(v_2) = 2 \\ \lambda(v_0 \wedge v_3) &= \lambda(v_0) + \lambda(v_3) = 0 & \lambda(v_1 \wedge v_2) &= \lambda(v_1) + \lambda(v_2) = 0 \\ \lambda(v_1 \wedge v_3) &= \lambda(v_1) + \lambda(v_3) = -2 & \lambda(v_2 \wedge v_3) &= \lambda(v_2) + \lambda(v_3) = -4. \end{aligned}$$

Note that $4, 2, 0, -2, -4$ are exactly the weights of V_4, and 0 is the (only) weight of V_0. Thus $\Lambda^2\ V_3 \cong V_4 \oplus V_0$. Correspondingly, $\mathrm{Sym}^2\ V_3 \cong V_6 \oplus V_2$.

References

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag New York Inc., New York, 1972.

[Rom08] Steven Roman. *Advanced Linear Algebra.* Springer, New York, 3rd edition, 2008.